

Chain rules and inequalities for the BHT fractional calculus on arbitrary time scales*

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Abstract

We develop the Benkhettou–Hassani–Torres fractional (noninteger order) calculus on time scales by proving two chain rules for the α -fractional derivative and five inequalities for the α -fractional integral. The results coincide with well-known classical results when the operators are of (integer) order $\alpha = 1$ and the time scale coincides with the set of real numbers.

Keywords: local fractional calculus; calculus on time scales; chain rules; integral inequalities.

MSC 2010: 26A33; 26D10; 26E70.

1 Introduction

The study of fractional (noninteger order) calculus on time scales is a subject of strong current interest [1, 2, 3, 4]. Recently, Benkhettou, Hassani and Torres introduced a (local) fractional calculus on arbitrary time scales \mathbb{T} (called here the BHT fractional calculus) based on the T_α differentiation operator and the α -fractional integral [5]. The Hilger time-scale calculus [6] is then obtained as a particular case, by choosing $\alpha = 1$. In this paper we develop the BHT time-scale fractional calculus initiated in [5]. Precisely, we prove two different chain rules for the fractional derivative T_α (Theorems 4 and 6) and several inequalities for the α -fractional integral: Hölder's inequality (Theorem 7), Cauchy–Schwarz's inequality (Theorem 8), Minkowski's inequality (Theorem 10), generalized Jensen's fractional inequality (Theorem 11) and a weighted fractional Hermite–Hadamard inequality on time scales (Theorem 12).

*This is a preprint of a paper whose final and definite form is published in *Arabian Journal of Mathematics*, ISSN 2193-5343 (Print) 2193-5351 (Online). Paper submitted 25/Feb/2016; revised 24/Sept/2016; accepted 28/Nov/2016.

The paper is organized as follows. In Section 2 we recall the basics of the the BHT fractional calculus. Our results are then formulated and proved in Section 3.

2 Preliminaries

We briefly recall the necessary notions from the BHT fractional calculus [5]: fractional differentiation and fractional integration on time scales. For an introduction to the time-scale theory we refer the reader to the book [6].

Definition 1 (See [5]). *Let \mathbb{T} be a time scale, $f : \mathbb{T} \rightarrow \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in (0, 1]$. For $t > 0$, we define $T_\alpha(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ -neighbourhood $\mathcal{V}_t = (t - \delta, t + \delta) \cap \mathbb{T}$ of t , $\delta > 0$, such that $|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$ for all $s \in \mathcal{V}_t$. We call $T_\alpha(f)(t)$ the α -fractional derivative of f of order α at t , and we define the α -fractional derivative at 0 as $T_\alpha(f)(0) := \lim_{t \rightarrow 0^+} T_\alpha(f)(t)$.*

If $\alpha = 1$, then we obtain from Definition 1 the Hilger delta derivative of time scales [6]. The α -fractional derivative of order zero is defined by the identity operator: $T_0(f) := f$. The basic properties of the α -fractional derivative on time scales are given in [5], together with several illustrative examples. Here we just recall the item (iv) of Theorem 4 in [5], which is needed in the proof of our Theorem 4.

Theorem 2 (See [5]). *Let $\alpha \in (0, 1]$ and \mathbb{T} be a time scale. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. If f is α -fractional differentiable of order α at t , then*

$$f(\sigma(t)) = f(t) + \mu(t)t^{\alpha-1}T_\alpha(f)(t).$$

The other main operator of [5] is the α -fractional integral of $f : \mathbb{T} \rightarrow \mathbb{R}$, defined by

$$\int f(t)\Delta^\alpha t := \int f(t)t^{\alpha-1}\Delta t,$$

where the integral on the right-hand side is the usual Hilger delta-integral of time scales [5, Def. 26]. If $F_\alpha(t) := \int f(t)\Delta^\alpha t$, then one defines the Cauchy α -fractional integral by $\int_a^b f(t)\Delta^\alpha t := F_\alpha(b) - F_\alpha(a)$, where $a, b \in \mathbb{T}$ [5, Def. 28]. The interested reader can find the basic properties of the Cauchy α -fractional integral in [5]. Here we are interested to prove some fractional integral inequalities on time scales. For that, we use some of the properties of [5, Theorem 31].

Theorem 3 (Cf. Theorem 31 of [5]). *Let $\alpha \in (0, 1]$, $a, b, c \in \mathbb{T}$, $\gamma \in \mathbb{R}$, and f, g be two rd-continuous functions. Then,*

- (i) $\int_a^b [f(t) + g(t)]\Delta^\alpha t = \int_a^b f(t)\Delta^\alpha t + \int_a^b g(t)\Delta^\alpha t;$
- (ii) $\int_a^b (\gamma f)(t)\Delta^\alpha t = \gamma \int_a^b f(t)\Delta^\alpha t;$

- (iii) $\int_a^b f(t) \Delta^\alpha t = - \int_b^a f(t) \Delta^\alpha t;$
- (iv) $\int_a^b f(t) \Delta^\alpha t = \int_a^c f(t) \Delta^\alpha t + \int_c^b f(t) \Delta^\alpha t;$
- (v) if there exist $g : \mathbb{T} \rightarrow \mathbb{R}$ with $|f(t)| \leq g(t)$ for all $t \in [a, b]$, then $\left| \int_a^b f(t) \Delta^\alpha t \right| \leq \int_a^b g(t) \Delta^\alpha t.$

3 Main Results

The chain rule, as we know it from the classical differential calculus, does not hold for the BHT fractional calculus. A simple example of this fact has been given in [5, Example 20]. Moreover, it has been shown in [5, Theorem 21], using the mean value theorem, that if $g : \mathbb{T} \rightarrow \mathbb{R}$ is continuous and fractional differentiable of order $\alpha \in (0, 1]$ at $t \in \mathbb{T}^\kappa$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, then there exists $c \in [t, \sigma(t)]$ such that $T_\alpha(f \circ g)(t) = f'(g(c))T_\alpha(g)(t)$. In Section 3.1, we provide two other chain rules. Then, in Section 3.2, we prove some fractional integral inequalities on time scales.

3.1 Fractional chain rules on time scales

Theorem 4 (Chain Rule I). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable, \mathbb{T} be a given time scale and $g : \mathbb{T} \rightarrow \mathbb{R}$ be α -fractional differentiable. Then, $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is also α -fractional differentiable with*

$$T_\alpha(f \circ g)(t) = \left[\int_0^1 f'(g(t) + h\mu(t)t^{\alpha-1}T_\alpha(g)(t)) dh \right] T_\alpha(g)(t). \quad (1)$$

Proof. We begin by applying the ordinary substitution rule from calculus:

$$\begin{aligned} f(g(\sigma(t))) - f(g(s)) &= \int_{g(s)}^{g(\sigma(t))} f'(\tau) d\tau \\ &= [g(\sigma(t)) - g(s)] \int_0^1 f'(hg(\sigma(t)) + (1-h)g(s)) dh. \end{aligned}$$

Let $t \in \mathbb{T}^\kappa$ and $\epsilon > 0$. Since g is α -fractional differentiable at t , we know from Definition 1 that there exists a neighbourhood U_1 of t such that

$$|[g(\sigma(t)) - g(s)]t^{1-\alpha} - T_\alpha(g)(t)(\sigma(t) - s)| \leq \epsilon^* |\sigma(t) - s| \quad \text{for all } s \in U_1,$$

where

$$\epsilon^* = \frac{\epsilon}{1 + 2 \int_0^1 |f'(hg(\sigma(t)) + (1-h)g(s))| dh}.$$

Moreover, f' is continuous on \mathbb{R} and, therefore, it is uniformly continuous on closed subsets of \mathbb{R} . Observing that g is also continuous, because it is α -fractional differentiable (see item (i) of Theorem 4 in [5]), there exists a neighbourhood U_2 of t such that

$$|f'(hg(\sigma(t)) + (1-h)g(s)) - f'(hg(\sigma(t)) + (1-h)g(t))| \leq \frac{\epsilon}{2(\epsilon^* + |T_\alpha(g)(t)|)}$$

for all $s \in U_2$. To see this, note that

$$\begin{aligned} |hg(\sigma(t)) + (1-h)g(s) - (hg(\sigma(t)) + (1-h)g(t))| &= (1-h)|g(s) - g(t)| \\ &\leq |g(s) - g(t)| \end{aligned}$$

holds for all $0 \leq h \leq 1$. We then define $U := U_1 \cap U_2$ and let $s \in U$. For convenience, we put

$$\gamma = hg(\sigma(t)) + (1-h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1-h)g(t).$$

Then we have

$$\begin{aligned} &\left| [(f \circ g)(\sigma(t)) - (f \circ g)(s)]t^{1-\alpha} - T_\alpha(g)(t)(\sigma(t) - s) \int_0^1 f'(\beta)dh \right| \\ &= \left| t^{1-\alpha}[g(\sigma(t)) - g(s)] \int_0^1 f'(\gamma)dh - T_\alpha(g)(t)(\sigma(t) - s) \int_0^1 f'(\beta)dh \right| \\ &= \left| \left(t^{1-\alpha}[g(\sigma(t)) - g(s)] - (\sigma(t) - s)T_\alpha(g)(t) \right) \right. \\ &\quad \left. \times \int_0^1 f'(\gamma)dh + T_\alpha(g)(t)(\sigma(t) - s) \int_0^1 (f'(\gamma) - f'(\beta))dh \right| \\ &\leq \left| t^{1-\alpha}[g(\sigma(t)) - g(s)] - (\sigma(t) - s)T_\alpha(g)(t) \right| \int_0^1 |f'(\gamma)|dh \\ &\quad + |T_\alpha(g)(t)| |\sigma(t) - s| \int_0^1 |f'(\gamma) - f'(\beta)|dh \\ &\leq \epsilon^* |\sigma(t) - s| \int_0^1 |f'(\gamma)|dh + [\epsilon^* + |T_\alpha(g)(t)|] |\sigma(t) - s| \int_0^1 |f'(\gamma) - f'(\beta)|dh \\ &\leq \frac{\epsilon}{2} |\sigma(t) - s| + \frac{\epsilon}{2} |\sigma(t) - s| \\ &= \epsilon |\sigma(t) - s|. \end{aligned}$$

Therefore, $f \circ g$ is α -fractional differentiable at t and (1) holds. \square

Let us illustrate Theorem 4 with an example.

Example 5. Let $g : \mathbb{Z} \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(t) = t^2 \text{ and } f(t) = e^t.$$

Then, $T_\alpha(g)(t) = (2t+1)t^{1-\alpha}$ and $f'(t) = e^t$. Hence, we have by Theorem 4 that

$$\begin{aligned}
T_\alpha(f \circ g)(t) &= \left[\int_0^1 f'(g(t) + h\mu(t)t^{\alpha-1}T_\alpha(g)(t))dh \right] T_\alpha(g)(t) \\
&= (2t+1)t^{1-\alpha} \int_0^1 e^{t^2+h(2t+1)}dh \\
&= (2t+1)t^{1-\alpha} e^{t^2} \int_0^1 e^{h(2t+1)}dh \\
&= (2t+1)t^{1-\alpha} e^{t^2} \frac{1}{2t+1} [e^{2t+1} - 1] \\
&= t^{1-\alpha} e^{t^2} [e^{2t+1} - 1].
\end{aligned}$$

Theorem 6 (Chain Rule II). *Let \mathbb{T} be a time scale. Assume $\nu : \mathbb{T} \rightarrow \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is also a time scale. Let $w : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$, $\alpha \in (0, 1]$, and \tilde{T}_α denote the α -fractional derivative on $\tilde{\mathbb{T}}$. If for each $t \in \mathbb{T}^\kappa$, $\tilde{T}_\alpha(w)(\nu(t))$ exists and for every $\epsilon > 0$, there is a neighbourhood V of t such that*

$$|\tilde{\sigma}(\nu(t)) - \nu(s) - T_\alpha(\nu)(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \quad \text{for all } s \in V,$$

where $\tilde{\sigma}$ denotes the forward jump operator on $\tilde{\mathbb{T}}$, then

$$T_\alpha(w \circ \nu)(t) = [\tilde{T}_\alpha(w) \circ \nu](t) T_\alpha(\nu)(t).$$

Proof. Let $0 < \epsilon < 1$ be given and define $\epsilon^* := \epsilon \left[1 + |T_\alpha(\nu)(t)| + |\tilde{T}_\alpha(w)(\nu(t))| \right]^{-1}$. Note that $0 < \epsilon^* < 1$. According to the assumptions, there exist neighbourhoods U_1 of t and U_2 of $\nu(t)$ such that

$$|\tilde{\sigma}(\nu(t)) - \nu(s) - T_\alpha(\nu)(t)(\sigma(t) - s)| \leq \epsilon^* |\sigma(t) - s|$$

for all $s \in U_1$ and

$$|[w(\tilde{\sigma}(\nu(t))) - w(r)]t^{1-\alpha} - \tilde{T}_\alpha(w)(\nu(t))(\tilde{\sigma}(\nu(t)) - r)| \leq \epsilon^* |\tilde{\sigma}(\nu(t)) - r|$$

for all $r \in U_2$. Let $U := U_1 \cap \nu^{-1}(U_2)$. For any $s \in U$, we have that $s \in U_1$ and $\nu(s) \in U_2$ with

$$\begin{aligned}
&\left| [w(\nu(\sigma(t))) - w(\nu(s))]t^{1-\alpha} - (\sigma(t) - s)[\tilde{T}_\alpha(w)(\nu(t))]T_\alpha(\nu)(t) \right| \\
&= \left| [w(\nu(\sigma(t))) - w(\nu(s))]t^{1-\alpha} - [\tilde{\sigma}(\nu(t)) - \nu(s)]\tilde{T}_\alpha(w)(\nu(t)) \right. \\
&\quad \left. + [\tilde{\sigma}(\nu(t)) - \nu(s) - T_\alpha(\nu)(t)(\sigma(t) - s)]\tilde{T}_\alpha(w)(\nu(t)) \right| \\
&\leq \epsilon^* |\tilde{\sigma}(\nu(t)) - \nu(s)| + \epsilon^* |\sigma(t) - s| |\tilde{T}_\alpha(w)(\nu(t))| \\
&\leq \epsilon^* \left[|\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)T_\alpha(\nu)(t)| \right. \\
&\quad \left. + |\sigma(t) - s| |T_\alpha(\nu)(t)| + |\sigma(t) - s| |\tilde{T}_\alpha(w)(\nu(t))| \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon^* \left[\epsilon^* |\sigma(t) - s| + |\sigma(t) - s| |T_\alpha(\nu)(t)| + |\sigma(t) - s| |\tilde{T}_\alpha(w)(\nu(t))| \right] \\
&= \epsilon^* |\sigma(t) - s| \left[\epsilon^* + |T_\alpha(\nu)(t)| + |\tilde{T}_\alpha(w)(\nu(t))| \right] \\
&\leq \epsilon^* \left[1 + |T_\alpha(\nu)(t)| + |\tilde{T}_\alpha(w)(\nu(t))| \right] |\sigma(t) - s| \\
&= \epsilon |\sigma(t) - s|.
\end{aligned}$$

This proves the claim. \square

3.2 Fractional integral inequalities on time scales

The α -fractional integral on time scales was introduced in [5, Section 3], where some basic properties were proved. Here we show that the α -fractional integral satisfies appropriate fractional versions of the fundamental inequalities of Hölder, Cauchy–Schwarz, Minkowski, Jensen and Hermite–Hadamard.

Theorem 7 (Hölder’s fractional inequality on time scales). *Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{T}$. If $f, g, h : [a, b] \rightarrow \mathbb{R}$ are rd-continuous, then*

$$\int_a^b |f(t)g(t)||h(t)|\Delta^\alpha t \leq \left[\int_a^b |f(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q |h(t)|\Delta^\alpha t \right]^{\frac{1}{q}}, \quad (2)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For nonnegative real numbers A and B , the basic inequality

$$A^{1/p} B^{1/q} \leq \frac{A}{p} + \frac{B}{q}$$

holds. Now, suppose, without loss of generality, that

$$\left[\int_a^b |f(t)|^p |h(t)|\Delta^\alpha t \right] \left[\int_a^b |g(t)|^q |h(t)|\Delta^\alpha t \right] \neq 0.$$

Applying Theorem 3 and the above inequality to

$$A(t) = \frac{|f(t)|^p |h(t)|}{\int_a^b |f(\tau)|^p |h(\tau)|\Delta^\alpha \tau}$$

and

$$B(t) = \frac{|g(t)|^q |h(t)|}{\int_a^b |g(\tau)|^q |h(\tau)|\Delta^\alpha \tau},$$

and integrating the obtained inequality between a and b , which is possible since

all occurring functions are rd -continuous, we find that

$$\begin{aligned}
& \int_a^b [A(t)]^{1/p} [B(t)]^{1/q} \Delta^\alpha t \\
&= \int_a^b \frac{|f(t)||h(t)|^{1/p}}{\left[\int_a^b |f(\tau)|^p |h(\tau)| \Delta^\alpha \tau \right]^{1/p}} \frac{|g(t)||h(t)|^{1/q}}{\left[\int_a^b |g(\tau)|^q |h(\tau)| \Delta^\alpha \tau \right]^{1/q}} \Delta^\alpha t \\
&\leq \int_a^b \left[\frac{A(t)}{p} + \frac{B(t)}{q} \right] \Delta^\alpha t \\
&= \int_a^b \left[\frac{1}{p} \frac{|f(t)|^p |h(t)|}{\int_a^b |f(\tau)|^p |h(\tau)| \Delta^\alpha \tau} + \frac{1}{q} \frac{|g(t)|^q |h(t)|}{\int_a^b |g(\tau)|^q |h(\tau)| \Delta^\alpha \tau} \right] \Delta^\alpha t \\
&= \frac{1}{p} \int_a^b \left[\frac{|f(t)|^p |h(t)|}{\int_a^b |f(\tau)|^p |h(\tau)| \Delta^\alpha \tau} \right] \Delta^\alpha t + \frac{1}{q} \int_a^b \left[\frac{|g(t)|^q |h(t)|}{\int_a^b |g(\tau)|^q |h(\tau)| \Delta^\alpha \tau} \right] \Delta^\alpha t \\
&\leq \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

This directly yields the Hölder inequality (2). \square

As a particular case of Theorem 7, we obtain the following inequality.

Theorem 8 (Cauchy–Schwarz’s fractional inequality on time scales). *Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{T}$. If $f, g, h : [a, b] \rightarrow \mathbb{R}$ are rd -continuous, then*

$$\int_a^b |f(t)g(t)||h(t)| \Delta^\alpha t \leq \sqrt{\left[\int_a^b |f(t)|^2 |h(t)| \Delta^\alpha t \right] \left[\int_a^b |g(t)|^2 |h(t)| \Delta^\alpha t \right]}.$$

Proof. Choose $p = q = 2$ in Hölder’s inequality (2). \square

Using Hölder’s inequality (2), we can also prove the following result.

Corollary 9. *Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{T}$. If $f, g, h : [a, b] \rightarrow \mathbb{R}$ are rd -continuous, then*

$$\int_a^b |f(t)g(t)||h(t)| \Delta^\alpha t \geq \left[\int_a^b |f(t)|^p |h(t)| \Delta^\alpha t \right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q |h(t)| \Delta^\alpha t \right]^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p < 0$ or $q < 0$.

Proof. Without loss of generality, we may assume that $p < 0$ and $q > 0$. Set $P = -\frac{p}{q}$ and $Q = \frac{1}{q}$. Then, $\frac{1}{P} + \frac{1}{Q} = 1$ with $P > 1$ and $Q > 0$. From (2) we

can write that

$$\begin{aligned} & \int_a^b |F(t)G(t)||h(t)|\Delta^\alpha t \\ & \leq \left[\int_a^b |F(t)|^P |h(t)|\Delta^\alpha t \right]^{\frac{1}{P}} \left[\int_a^b |G(t)|^Q |h(t)|\Delta^\alpha t \right]^{\frac{1}{Q}} \end{aligned} \quad (3)$$

for any rd -continuous functions $F, G : [a, b] \rightarrow \mathbb{R}$. The desired result is obtained by taking $F(t) = [f(t)]^{-q}$ and $G(t) = [f(t)]^q [g(t)]^q$ in inequality (3). \square

Next, we use Hölder's inequality (2) to deduce a fractional Minkowski's inequality on time scales.

Theorem 10 (Minkowski's fractional inequality on time scales). *Let $\alpha \in (0, 1]$, $a, b \in \mathbb{T}$ and $p > 1$. If $f, g, h : [a, b] \rightarrow \mathbb{R}$ are rd -continuous, then*

$$\begin{aligned} & \left[\int_a^b |(f+g)(t)|^p |h(t)|\Delta^\alpha t \right]^{1/p} \\ & \leq \left[\int_a^b |f(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} + \left[\int_a^b |g(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}}. \end{aligned} \quad (4)$$

Proof. We apply Hölder's inequality (2) with $q = p/(p-1)$ and items (i) and (v) of Theorem 3 to obtain

$$\begin{aligned} & \int_a^b |(f+g)(t)|^p |h(t)|\Delta^\alpha t \\ & = \int_a^b |(f+g)(t)|^{p-1} |(f+g)(t)| |h(t)|\Delta^\alpha t \\ & \leq \int_a^b |f(t)| |(f+g)(t)|^{p-1} |h(t)|\Delta^\alpha t + \int_a^b |g(t)| |(f+g)(t)|^{p-1} |h(t)|\Delta^\alpha t \\ & \leq \left[\int_a^b |f(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \left[\int_a^b |(f+g)(t)|^{(p-1)q} |h(t)|\Delta^\alpha t \right]^{\frac{1}{q}} \\ & \quad + \left[\int_a^b |g(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \left[\int_a^b |(f+g)(t)|^{(p-1)q} |h(t)|\Delta^\alpha t \right]^{\frac{1}{q}} \\ & = \left[\int_a^b |(f+g)(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{q}} \\ & \quad \times \left(\left[\int_a^b |f(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} + \left[\int_a^b |g(t)|^p |h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \right). \end{aligned}$$

Dividing both sides of the obtained inequality by $\left[\int_a^b |(f+g)(t)|^p |h(t)| \Delta^\alpha t \right]^{\frac{1}{q}}$, we arrive at the Minkowski inequality (4). \square

Jensen's classical inequality relates the value of a convex/concave function of an integral to the integral of the convex/concave function. We prove a generalization of such relation for the BHT fractional calculus on time scales.

Theorem 11 (Generalized Jensen's fractional inequality on time scales). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $c, d \in \mathbb{R}$, $\alpha \in (0, 1]$, $g \in C([a, b] \cap \mathbb{T}; (c, d))$ and $h \in C([a, b] \cap \mathbb{T}; \mathbb{R})$ with*

$$\int_a^b |h(s)| \Delta^\alpha s > 0.$$

- If $f \in C((c, d); \mathbb{R})$ is convex, then

$$f \left(\frac{\int_a^b g(s) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s} \right) \leq \frac{\int_a^b f(g(s)) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s}. \quad (5)$$

- If $f \in C((c, d); \mathbb{R})$ is concave, then

$$f \left(\frac{\int_a^b g(s) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s} \right) \geq \frac{\int_a^b f(g(s)) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s}. \quad (6)$$

Proof. We start by proving (5). Since f is convex, for any $t \in (c, d)$ there exists $a_t \in \mathbb{R}$ such that

$$a_t(x - t) \leq f(x) - f(t) \quad \text{for all } x \in (c, d). \quad (7)$$

Let

$$t = \frac{\int_a^b g(s) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s}.$$

It follows from (7) and item (v) of Theorem 3 that

$$\begin{aligned} & \int_a^b f(g(s)) |h(s)| \Delta^\alpha s - \left(\int_a^b |h(s)| \Delta^\alpha s \right) f \left(\frac{\int_a^b g(s) |h(s)| \Delta^\alpha s}{\int_a^b |h(s)| \Delta^\alpha s} \right) \\ &= \int_a^b f(g(s)) |h(s)| \Delta^\alpha s - \left(\int_a^b |h(s)| \Delta^\alpha s \right) f(t) \\ &= \int_a^b (f(g(s)) - f(t)) |h(s)| \Delta^\alpha s \end{aligned}$$

$$\begin{aligned}
&\geq a_t \int_a^b (g(s) - t) |h(s)| \Delta^\alpha s \\
&= a_t \left(\int_a^b g(s) |h(s)| \Delta^\alpha s - t \int_a^b |h(s)| \Delta^\alpha s \right) \\
&= a_t \left(\int_a^b g(s) |h(s)| \Delta^\alpha s - \int_a^b g(s) |h(s)| \Delta^\alpha s \right) \\
&= 0.
\end{aligned}$$

This proves (5). To prove (6), we simply observe that $F(x) = -f(x)$ is convex (because we are now assuming f to be concave) and then we apply inequality (5) to function F . \square

We end with an application of Theorem 11.

Theorem 12 (A weighted fractional Hermite–Hadamard inequality on time scales). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ and $\alpha \in (0, 1]$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous convex function and let $w : \mathbb{T} \rightarrow \mathbb{R}$ be a continuous function such that $w(t) \geq 0$ for all $t \in \mathbb{T}$ and $\int_a^b w(t) \Delta^\alpha t > 0$. Then,*

$$f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t) \Delta^\alpha t} \int_a^b f(t) w(t) \Delta^\alpha t \leq \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b), \quad (8)$$

$$\text{where } x_{w,\alpha} = \frac{\int_a^b t w(t) \Delta^\alpha t}{\int_a^b w(t) \Delta^\alpha t}.$$

Proof. For every convex function one has

$$f(t) \leq f(a) + \frac{f(b) - f(a)}{b - a} (t - a).$$

Multiplying this inequality with $w(t)$, which is nonnegative, we get

$$w(t) f(t) \leq f(a) w(t) + \frac{f(b) - f(a)}{b - a} (t - a) w(t).$$

Taking the α -fractional integral on both sides, we can write that

$$\int_a^b w(t) f(t) \Delta^\alpha t \leq \int_a^b f(a) w(t) \Delta^\alpha t + \int_a^b \frac{f(b) - f(a)}{b - a} (t - a) w(t) \Delta^\alpha t,$$

which implies

$$\begin{aligned}
&\int_a^b w(t) f(t) \Delta^\alpha t \\
&\leq f(a) \int_a^b w(t) \Delta^\alpha t + \frac{f(b) - f(a)}{b - a} \left(\int_a^b t w(t) \Delta^\alpha t - a \int_a^b w(t) \Delta^\alpha t \right),
\end{aligned}$$

that is,

$$\frac{1}{\int_a^b w(t) \Delta^\alpha t} \int_a^b f(t) w(t) \Delta^\alpha t \leq \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b).$$

We have just proved the second inequality of (8). For the first inequality of (8), we use (5) of Theorem 11 by taking $g : \mathbb{T} \rightarrow \mathbb{T}$ defined by $g(s) = s$ for all $s \in \mathbb{T}$ and $h : \mathbb{T} \rightarrow \mathbb{R}$ given by $h = w$. \square

Note that if in Theorem 12 we consider a concave function f instead of a convex one, then the inequalities of (8) are reversed.

Acknowledgements

Torres was partially supported by the Portuguese Foundation for Science and Technology (FCT), through the Center for Research and Development in Mathematics and Applications (CIDMA), within project UID/MAT/04106/2013. The authors are greatly indebted to two referees for their several useful suggestions and valuable comments.

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